From a Binomial Model for the Value of Stock to Option Pricing by Black–Scholes–Merton

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Goals of this Presentation

- Have a closer look at the daily changes of stock values.
- Use a simple binomial model to describe the evolution of stock values.
- Derive a finite difference approximation for a PDE modelling the evolution of the values of a stock.
- Give a very brief explanation of options.
- Derive the Black–Scholes PDE for the values of European put or call options.

Key point: All based on elementary algebra, some analysis and probability, i.e. no advanced mathematical tools.

Market Observations I

Examine historical data of stock values, in this case the daily values for the years 2015–2019. Use $S(i)$, the value on the i–th day of trading. Use relative changes (e.g. $+0.5\%$) and determine

$$\text{daily gain}(i) = \log(\frac{S(i)}{S(i-1)})$$

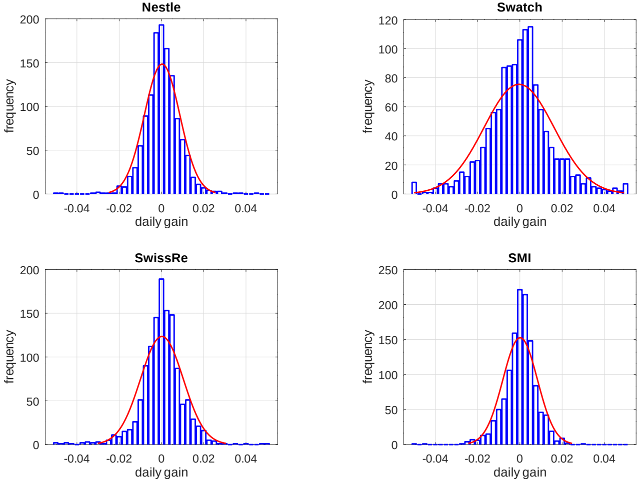
The value on the N–th day of trading is given by

$$S(N) = S(0) \cdot \exp(\sum_{i=1}^{N} \text{daily gain}(i))$$

Examine histograms of daily gains for a few stocks.

Market Observations II

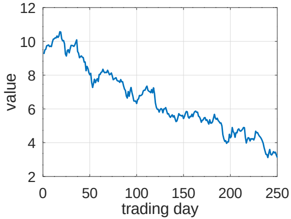
Historical data based on five years from 2015 until 2019.

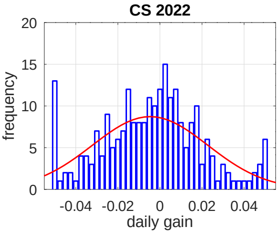


Market Observations III

A normal distribution is often a very good assumption, but too many extreme events occur. As example consider the results for the Credit Swiss stock in 2022.

- annual gain -69%, -0.49% per day
- annual volatility 50.4%, 3.25% per day





Observe the wider spread of the daily gains and the large number of extreme events.

Market Observations IV

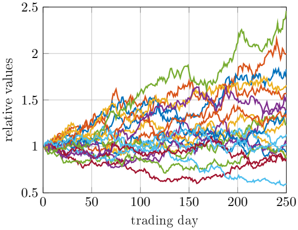
Based on the data estimate the daily gain and its standard deviation. This allows to determine an annual gain and the corresponding standard deviation, called **volatility**, of the value of the stock.

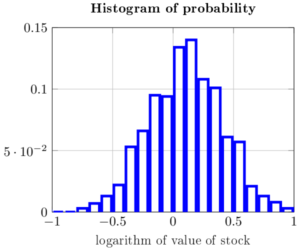
	Nestle	Swatch	SwissRe	SMI
daily gain	+0.031%	-0.039%	+0.021%	+0.013%
annual gain r	+7.609%	-8.867%	+5.189%	+3.282%
daily stddev	0.842%	1.656%	1.012%	0.819%
volatility σ	13.05%	25.65%	15.67%	12.69%

To create a model for the development of the value $S(t)$ of a stock **assume** a known annual gain r (resp. $e^r - 1$) and a volatility σ. This will allow to determine the value V of an option based on this stock.

Market Observations V

The histograms of daily gains look (almost) like normal distributions. Use Monte Carlo simulations to model future values of a stock. Based on data from 1990 to 1999 for IBM stock estimate the daily gain at 0.0605% and its standard deviation at 1.94%.





This is a possible tool to determine the value of an option based on this specific stock.

An (overly?) simple model:

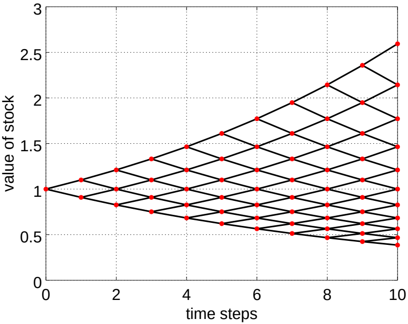
Examine a stock with values $S(t)$, resp. $S(i\,\Delta t)$. For each time step Δt (e.g. one day) the value will

- either be multiplied by $e^{+u} > 1$ with probability $\frac{1}{2}\,(1+\Delta p)$
- or be multiplied by $e^{-u} < 1$ with probability $\frac{1}{2}\,(1-\Delta p)$

$$\begin{aligned}
S_1 &= S(1\,\Delta t) = \begin{cases} S_0\,e^{+u} & \text{with probability } \frac{1}{2}\,(1+\Delta p) \\ S_0\,e^{-u} & \text{with probability } \frac{1}{2}\,(1-\Delta p) \end{cases} \\
S_2 &= S(2\,\Delta t) = \begin{cases} S_1\,e^{+u} & \text{with probability } \frac{1}{2}\,(1+\Delta p) \\ S_1\,e^{-u} & \text{with probability } \frac{1}{2}\,(1-\Delta p) \end{cases} \\
S_3 &= S(3\,\Delta t) = \begin{cases} S_2\,e^{+u} & \text{with probability } \frac{1}{2}\,(1+\Delta p) \\ S_2\,e^{-u} & \text{with probability } \frac{1}{2}\,(1-\Delta p) \end{cases} \\
&\vdots
\end{aligned}$$

Binomial Model II

For a starting value $S_0 = 1$ this can be visualized by a mesh with a linear scale for the values $S_i = S(i\,\Delta t)$.

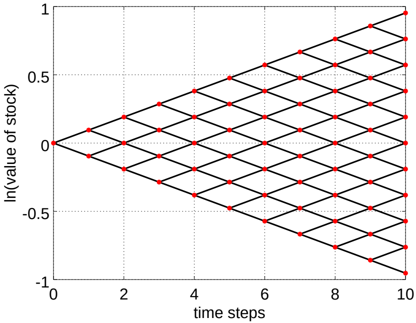


Binomial Model III

For easier calculations use a logarithmic scale $z(t) = \ln(S(t))$.

$$\begin{aligned}
z_1 &= \begin{cases} z_0 + u & \text{with probability } \frac{1}{2}(1+\Delta p) \\ z_0 - u & \text{with probability } \frac{1}{2}(1-\Delta p) \end{cases} \\
z_2 &= \begin{cases} z_1 + u & \text{with probability } \frac{1}{2}(1+\Delta p) \\ z_1 - u & \text{with probability } \frac{1}{2}(1-\Delta p) \end{cases} \\
z_3 &= \begin{cases} z_2 + u & \text{with probability } \frac{1}{2}(1+\Delta p) \\ z_2 - u & \text{with probability } \frac{1}{2}(1-\Delta p) \end{cases} \\
&\vdots
\end{aligned}$$

Binomial Model IV



Of the n steps k went upwards, i.e. find a binomial distribution.

$$\begin{aligned} P(z_n = z_0 + (2\,k - n)\,u) &= \binom{n}{k}\,\frac{1}{2^n}\,(1+\Delta p)^k\,(1-\Delta p)^{n-k} \\ &= \mathrm{PDF}(k, \mathrm{B}(n, \frac{1}{2}(1+\Delta p))) \end{aligned}$$

E and σ^2 for a Binomial Distribution I

Since the expected value E and the variance σ^2 of a binomial distribution will be used, here an elementary, elegant derivation:

- For the expected value E use the standard binomial formula with $q = 1 - p$. Differentiate once with respect to p, then multiply by p.

$$\begin{aligned}
(p+q)^n &= \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} \\
p\, n\, (p+q)^{n-1} &= p\, \frac{d}{dp}\, (p+q)^n = \sum_{i=0}^{n} i \binom{n}{i} p^i q^{n-i} \\
p\, n &= \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i} = E(X)
\end{aligned}$$

E and σ^2 for a Binomial Distribution II

- For the variance σ^2 differentiate twice, multiply by p^2.

$$\begin{aligned}
p^2 n\,(n-1)\,(p+q)^{n-2} &= p^2\,\frac{d^2}{dp^2}\,(p+q)^n \\
&= \sum_{i=0}^{n} i\,(i-1)\binom{n}{i} p^i q^{n-i} \\
p^2 n\,(n-1) &= \sum_{i=0}^{n} i\,(i-1)\binom{n}{i} p^i (1-p)^{n-i} \\
&= E(X^2) - E(X) = E(X^2) - n\,p \\
E(X^2) &= p^2 n\,(n-1) + n\,p = n\,p\,(n\,p - p + 1) \\
\sigma^2 &= E(X^2) - E(X)^2 \\
&= n\,p\,(n\,p - p + 1) - n^2 p^2 = n\,p\,(1-p)
\end{aligned}$$

E and σ^2 for a Binomial Distribution III

Determine $E(z_n)$ and $\mathrm{Var}(z_n)$ for the binomial distribution $\mathrm{B}(n, \frac{1}{2}(1+\Delta p))$ for the values of the stock. Use $p = \frac{1}{2}\,(1+\Delta p)$.

$$\begin{aligned}
\mathrm{E}(z_n) &= \sum_{k=0}^{n} (z_0 + (2\,k - n)\;u)\;\;\mathrm{PDF}(k, \mathrm{B}(n, \frac{1}{2}(1+\Delta p))) \\
&= (z_0 - n\,u)\;\mathrm{E}(1) + 2\,u\;\mathrm{E}(k) \\
&= z_0 - n\,u + u\,n\,(1+\Delta p) \\
&= z_0 + n\,u\,\Delta p
\end{aligned}$$

E and σ^2 for a Binomial Distribution IV

$$
\begin{aligned}
\mathrm{Var}(z_n) &= \mathrm{E}((z_n - \mathrm{E}(z_n))^2) \\
&= \sum_{k=0}^{n} ((2\,k-n)\,u - n\,u\,\Delta p)^2\ \mathrm{PDF}(k, \mathrm{B}(n, \frac{1}{2}(1+\Delta p))) \\
&= u^2 \sum_{k=0}^{n} (4\,k^2 - 2\,k\,n + n^2 - 2\,(2\,k-n)\,n\,\Delta p + n^2\,(\Delta p)^2) \cdot \\
&\qquad\qquad \cdot \mathrm{PDF}(k, \mathrm{B}(n, \frac{1}{2}(1+\Delta p))) \\
&= u^2\,(4\ \mathrm{E}(k^2) - 4\,n\,(1+\Delta p)\ \mathrm{E}(k) + n^2 (1+\Delta p)^2\ \mathrm{E}(1)) \\
&= u^2 \Big(4\,n\,\frac{1}{2}\,(1+\Delta p)\ \frac{1}{2}\ (1 - \Delta p + n\,(1+\Delta p)) - \\
&\qquad -4\,n\,(1+\Delta p)\ n\ \frac{1}{2}\,(1+\Delta p) + n^2\,(1+\Delta p)^2\Big) \\
&= \cdots \\
&= u^2\,n\,(1-(\Delta p)^2)
\end{aligned}
$$

Compatibility Condition and Parameter Selection I

Take n time steps of equal length $\Delta t = \frac{1}{n}$ to reach time $t = 1$. Based on the market observations ask that the expectation value $E(z_n)$ and the variance σ^2 satisfy

$$\begin{aligned} z_0 + r &= E(z_n) = z_0 + n\,u\,\Delta p \\ \sigma^2 &= V(z_n) = u^2\,n\,(1 - (\Delta p)^2) \end{aligned}$$

Solve the two equations for the parameters Δp and u.

$$\begin{aligned} \Delta p = \frac{r}{n\,u} \quad &\Longrightarrow \quad \sigma^2 = u^2\,n\,(1 - \frac{r^2}{n^2\,u^2}) = n\,u^2 - \frac{r^2}{n} \\ &\Longrightarrow \quad u = \frac{\sqrt{n\,\sigma^2 + r^2}}{n} \\ &\Longrightarrow \quad \Delta p = \frac{r}{n\,u} = \frac{r}{\sqrt{n\,\sigma^2 + r^2}} \end{aligned}$$

Compatibility Condition and Parameter Selection II

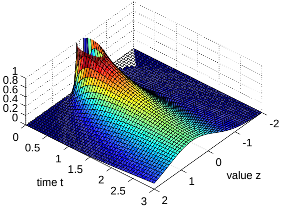
For $n \gg 1$ (with $t = 1$) find the approximations

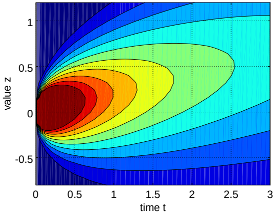
$$\Delta p = \frac{r}{n\,u} \approx \frac{r}{\sigma\,\sqrt{n}} \quad \text{and} \quad u \approx \frac{\sigma}{\sqrt{n}}\ .$$

Using the De Moivre–Laplace theorem with these parameters and $n \to \infty$ leads to a normal distribution at time t, given by

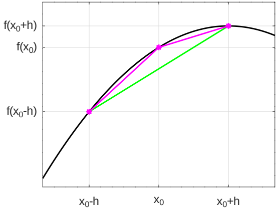
$$\mathrm{PDF}(t,z) = \frac{1}{\sigma\,\sqrt{2\,\pi}\,\sqrt{t}}\; e^{-(z-(z_0+t\,r))^2/(2\,t\,\sigma^2)}$$

with mean $z_0 + r\,t$ and variance $t\,\sigma^2$.





Finite Difference Approximations of Derivatives



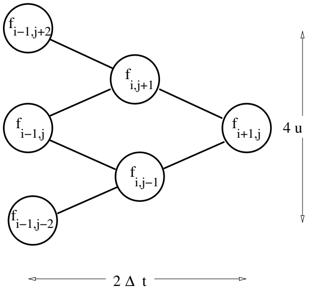
$$\begin{aligned}
f'(x_0) &\approx \frac{f(x_0+h)-f(x_0-h)}{2\,h} \\
f''(x_0) &\approx \frac{f'(x_0+h/2)-f'(x_0-h/2)}{h} \\
&\approx \frac{1}{h}\left(\frac{f(x_0+h)-f(x_0)}{h}-\frac{f(x_0)-f(x_0-h)}{h}\right) \\
&= \frac{f(x_0-h)-2\,f(x_0)+f(x_0+h)}{h^2}
\end{aligned}$$

From the Binomial Model to a PDE I

The goal is to verify that the above binomial model leads to the finite difference discretization of a PDE (Partial Differential Equation). Use

$$f_{i,j} = \text{value of probability at } t = i\,\Delta t \text{ and } z = j \cdot u$$

and the stencil



From the Binomial Model to a PDE II

Applying the formula for the binomial tree

$$f_{i+1,j} = \frac{1}{2}\,(1+\Delta p)\,f_{i,j-1} + \frac{1}{2}\,(1-\Delta p)\,f_{i,j+1}$$

repeatedly with $u = \frac{\sigma}{\sqrt{n}}$, $\Delta p = \frac{r}{\sigma\,\sqrt{n}} = \frac{r}{n\,u}$ and elementary algebra leads to

$$\begin{aligned} 2\,f_{i,j-1} &= (1+\Delta p)\,f_{i-1,j-2} + (1-\Delta p)\,f_{i-1,j} \\ 2\,f_{i,j+1} &= (1+\Delta p)\,f_{i-1,j} + (1-\Delta p)\,f_{i-1,j+2} \end{aligned}$$

From the Binomial Model to a PDE III

$$
\begin{aligned}
4\, f_{i+1,j} &= (1+\Delta p)\, 2\, f_{i,j-1} + (1-\Delta p)\, 2\, f_{i,j+1} \\
&= (1+\Delta p)\ \left((1+\Delta p)\, f_{i-1,j-2} + (1-\Delta p)\, f_{i-1,j}\right) \\
&\quad +(1-\Delta p)\ \left((1+\Delta p)\, f_{i-1,j} + (1-\Delta p)\, f_{i-1,j+2}\right) \\
&= (1+\Delta p)^2\ f_{i-1,j-2} + 2\,(1-(\Delta p)^2)\ f_{i-1,j} + (1-\Delta p)^2\ f_{i-1,j+2} \\
&= (1+\frac{r}{n\,u})^2\ f_{i-1,j-2} + 2\,(1-\frac{r^2}{n^2\,u^2})\ f_{i-1,j} + (1-\frac{r}{n\,u})^2\ f_{i-1,j+2} \\
&= (f_{i-1,j-2} + 2\, f_{i-1,j} + f_{i-1,j+2}) + \frac{2\,r\,(f_{i-1,j-2} - f_{i-1,j+2})}{n\,u} + \\
&\quad + \frac{r^2\,(f_{i-1,j-2} - 2\, f_{i-1,j} + f_{i-1,j+2})}{n^2\,u^2}
\end{aligned}
$$

From the Binomial Model to a PDE IV

Use $\Delta t = \frac{1}{n} = \frac{u^2}{\sigma^2}$ and the limit as $n \to \infty$, or $\Delta t \to 0$ and $u \to 0$.

$$\begin{aligned}
4\,\frac{f_{i+1,j} - f_{i-1,j}}{\Delta t} &= \sigma^2\,\frac{f_{i-1,j-2} - 2\,f_{i-1,j} + f_{i-1,j+2}}{u^2} + 2\,r\,\frac{f_{i-1,j-2} - f_{i-1,j+2}}{u} + \\
&\quad + \frac{r^2}{n}\,\frac{f_{i-1,j-2} - 2\,f_{i-1,j} + f_{i-1,j+2}}{u^2} \\
\frac{f_{i+1,j} - f_{i-1,j}}{2\,\Delta t} &= \frac{\sigma^2}{2}\,\frac{f_{i-1,j-2} - 2\,f_{i-1,j} + f_{i-1,j+2}}{4\,u^2} - r\,\frac{f_{i-1,j+2} - f_{i-1,j-2}}{4\,u} + \\
&\quad + \frac{r^2}{2\,n}\,\frac{f_{i-1,j-2} - 2\,f_{i-1,j} + f_{i-1,j+2}}{4\,u^2}
\end{aligned}$$

If $f_{i,j}$ is the discretization of a continuous function $f(t,z)$, i.e. $f_{i,j} = f(i\,\Delta t, j\,\Delta z)$, take the limit $n \to \infty$ and recognize the above as an explicit finite difference approximation of the dynamic heat equation

$$\frac{\partial f(t,z)}{\partial t} = \frac{\sigma^2}{2}\,\frac{\partial^2 f(t,z)}{\partial z^2} - r\,\frac{\partial f(t,z)}{\partial z} + 0\,.$$

The above function $\mathrm{PDF}(t,z) = f(t,z)$ satisfies this PDE.

From the Binomial Model to a PDE V

Thus the probability distribution of the value of a stock satisfies the same PDE as a heat equation with some drift contribution

$$\frac{\partial\,\mathrm{PDF}(t,z)}{\partial t} = \frac{\sigma^2}{2}\,\frac{\partial^2\,\mathrm{PDF}(t,z)}{\partial z^2} - r\,\frac{\partial\,\mathrm{PDF}(t,z)}{\partial z}$$

with time t and $z = \ln(S)$, where S is the value of the stock. This is a PDE with constant coefficients. Rewriting this with the stock value $S = e^z$ as independent variable with $V(t,S) = \mathrm{PDF}(t,\ln(S))$ leads to

$$\frac{\partial\,V(t,S)}{\partial t} = \frac{\sigma^2}{2}\,S^2\,\frac{\partial^2\,V(t,S)}{\partial S^2} + (\frac{\sigma^2}{2} - r)\,S\,\frac{\partial\,V(t,S)}{\partial S}\,.$$

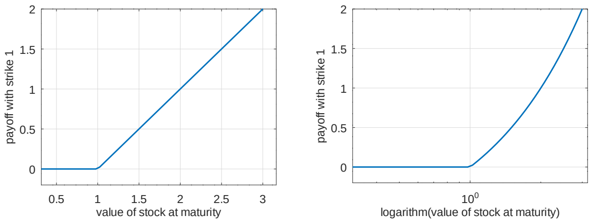
This is a PDE with variable coefficients. Often this form is used in the literature!

What is an Option?

As holder of an **European call option** you have the right, but not the obligation, to buy the underlying asset (e.g. stock, currency) at a given maturity date T for the given strike price K. For this right you have to pay a price V, the value of the option.

On the maturity date T there are different outcomes possible:

- If at time T the value S of the stock is below the strike price K, do nothing.
- If at time T the value S of the stock is above the strike K, call your option, and sell the stock on the market, you gain $S - K$.



Different Types of Options

There are many different types of options.

- European call option: right to buy at maturity date
- European put option: right to sell at maturity date
- American call option: right to buy at any time before the maturity date
- American put option: right to sell at any time before the maturity date
- Different payoffs are possible, e.g. binary
- many, many more, some incredibly complicated

Options: Insurance or Speculation? I

A put option with a stock as underlying asset:

- **Insurance**: you own a large chunk of Swisscom stock and want to use it a year from now to pay off your mortage, since you retire. To make sure that you obtain a minimal price for your Swisscom stock, you buy a put option with your minimal price as strike. You have to buy this insurance at a fair price.
- **Speculation**: your gut feeling (or insider knowledge) tells you that the value of the Swisscom stock will fall drastically within a year. With a put option you assure that you can sell at the strike price. If a year from now the value is considerably lower than your strike, you buy on the market and make a (hopefully large) profit. You risk to loose the price of this option, bought at a fair price.

Options: Insurance or Speculation? II

A call option with a currency exchange as underlying asset:

- **Insurance**: as a small business owner you have a large contract in the USA and your work will be paid a year from now with 1 million USD. With the help of a call option for CHF (payed in USD) with a strike of 0.90 $\frac{\text{CHF}}{\text{USD}}$ you are assured to have at least this "exchange rate", even if the value of the USD would fall drastically. You have to buy this insurance at a fair price.
- **Speculation**: your gut feeling (or insider knowledge) tells you that the value of USD will fall drastically within a year. Thus you buy a call option for CHF at a strike of 0.90 $\frac{\text{CHF}}{\text{USD}}$ for 1 million USD at a fair price. If the exchange rate on the market a year from now would be 0.80 $\frac{\text{CHF}}{\text{USD}}$ you buy 1'000'000 USD for 800'000 CHF on the market and then obtain 900'000 CHF by calling the option. It would be a nice gain of 100'000 CHF, but you risk to loose the price of the option.

Fair Value of an Option

The key point is to determine the fair value V of an option, as function of time and the current value S_0 of the underlying asset:

The fair value equals the expected payoff

Assume that there is a save interest rate r_0 (currently rather small). For a call option with payoff $\max\{0, S - K\} = \max\{0, e^z - K\}$ this leads to

$$\begin{aligned} V(t, z_0)\, e^{+r_0(T-t)} &= \int_{-\infty}^{+\infty} \max\{0, e^{\tilde{z}} - K\} \cdot \mathrm{PDF}(T, z)\, dz \\ &= \int_{\ln K}^{+\infty} (e^z - K) \cdot \mathrm{PDF}(T, z)\, dz \end{aligned}$$

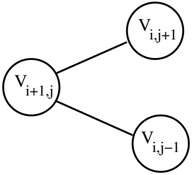
and based on this generate explicit formulas for the value $V(t, z_0)$ for European call or put options at time t with current value $S_0 = e^{z_0}$.

From the Binomial Model to Black–Scholes I

Apply the same idea for the binomial model. Use the notations

$$V_{i,j} = \text{value of option at } t = T - i\,\Delta t \text{ and } z = j \cdot u$$

and the risk-less interest rate r_0 and the "fair value condition".



$$\begin{aligned} \text{current value} + \text{interest} &= \text{expected future value} \\ e^{r_0/n} \cdot V_{i+1,j} &= \frac{1}{2}\,(1-\Delta p)\,V_{i,j-1} + \frac{1}{2}\,(1+\Delta p)\,V_{i,j+1} \end{aligned}$$

From the Binomial Model to Black–Scholes II

With $f_{i,j} = e^{i\, r_0/n} \cdot V_{i,j}$ this implies

$$f_{i+1,j} = \frac{1}{2}\,(1-\Delta p)\, f_{i,j-1} + \frac{1}{2}\,(1+\Delta p)\, f_{i,j+1}$$

i.e. the same difference equation as for the value z of the stock, but with a change of sign in $\Delta p = \frac{r}{2}\,\frac{1}{n\,u}$. Thus the modified function $f(t,z) = e^{r_0(T-t)} \cdot V(T-t,z)$ satisfies the PDE

$$\frac{\partial f(t,z)}{\partial t} = \frac{\sigma^2}{2}\,\frac{\partial^2 f(t,z)}{\partial z^2} + r\,\frac{\partial f(t,z)}{\partial z}$$

From the Binomial Model to Black–Scholes III

For the original function $V(\tau,z) = e^{r_0\,\tau}\,f(T-\tau)$ conclude

$$\begin{aligned}\frac{\partial}{\partial\tau}\,V(\tau,z) &= r_0\,e^{r_0\,\tau}\,f(T-\tau) - e^{r_0\,\tau}\left(\frac{\sigma^2}{2}\,\frac{\partial^2 f(t,z)}{\partial z^2} + r\,\frac{\partial f(t,z)}{\partial z}\right)\\ &= r_0\,V(\tau,z) - \frac{\sigma^2}{2}\,\frac{\partial^2}{\partial z^2}\,V(\tau,z) - r\,\frac{\partial}{\partial z}\,V(\tau,z)\end{aligned}$$

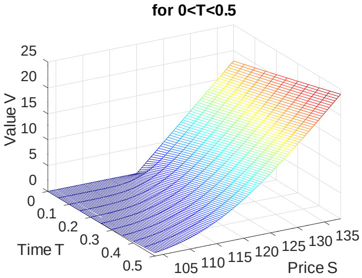
and consequently $V(t,z)$ solves the PDE

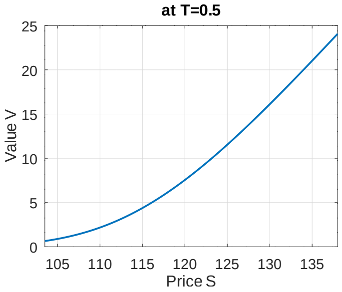
$$-\frac{\partial V(t,z)}{\partial t} = \frac{\sigma^2}{2}\,\frac{\partial^2 V(t,z)}{\partial z^2} + r\,\frac{\partial V(t,z)}{\partial z} - r_0\,V(t,z)$$

This is the **Black–Scholes–Merton** PDE, leading to a Nobel Memorial Price in 1997. It is similar to a dynamic heat equation with time reversal and the final condition $V(T,z) = \text{payoff}(z)$.

Value of a Call Option on Nestle Stock

Use the historic data for Nestle stock from 2015 until 2019 with a strike of $K = 120$. T is the time until maturity of the call option. Use *Octave* and implement a finite difference scheme to solve the PDE and determine the value V of the option as function of time T and price S of the stock.





American Options

With an American call option you have to right to call the option at any time $t \leq T$. Thus an American options is always more valuable than an European option. The resulting PDE is again

$$-\frac{\partial V(t,z)}{\partial t} = \frac{\sigma^2}{2}\,\frac{\partial^2 V(t,z)}{\partial z^2} + r\,\frac{\partial V(t,z)}{\partial z} - r_0\,V(t,z)\;,$$

but with the additional obstacle

$$V(t,z) \geq \text{payoff}(z)\;.$$

If $V(t,z) < \text{payoff}(z)$ then you would cash in immediately.

This PDE is considerably more difficult to derive and solve numerically. It is a nonlinear problem, caused by the obstacle. There are no analytical solutions.

Some Literature

- Find the historical data at `finance.yahoo.com/`
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Thank You for Your Attention

That's all folks