Some financial applications of Kalman filtering

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15 septembre 2012

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How to make it better ?

ARMAX ?

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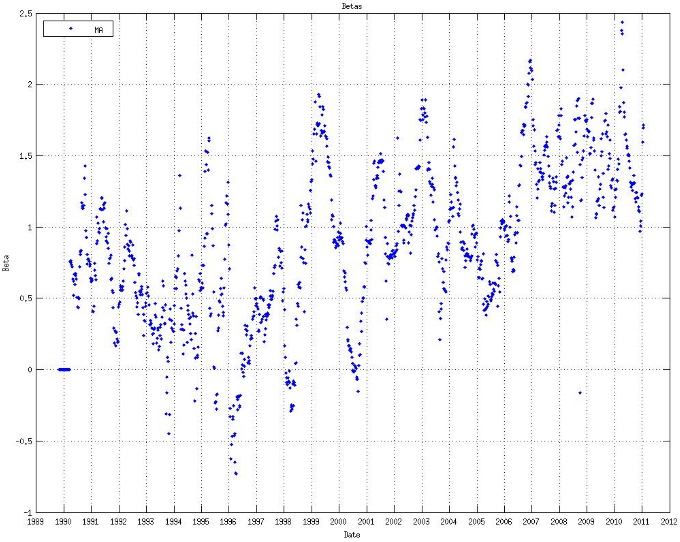
Dynamic ?

$$\begin{aligned} y_t &= x_t \beta_t + \epsilon_t,\ \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2) \\ \beta_t &= T \beta_{t-1} + \eta_t,\ \eta_t \sim \mathcal{N}(0, \sigma_\eta^2) \end{aligned}$$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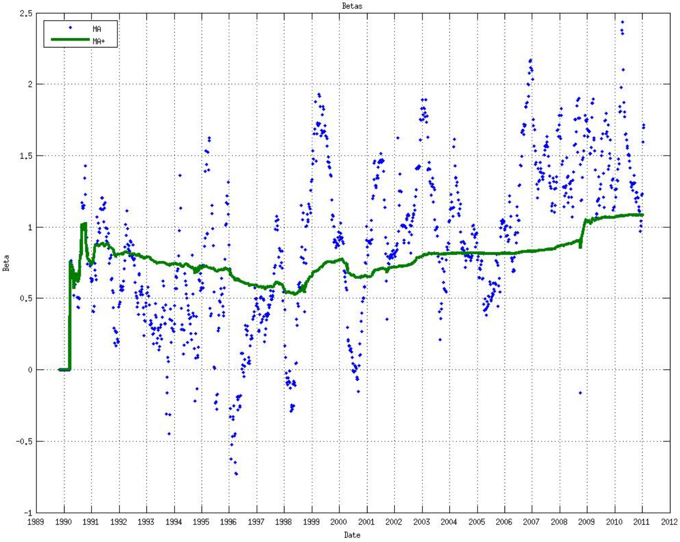
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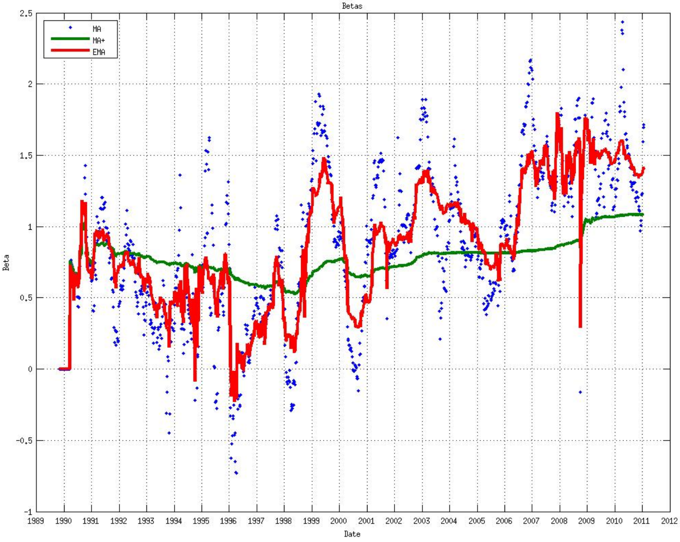
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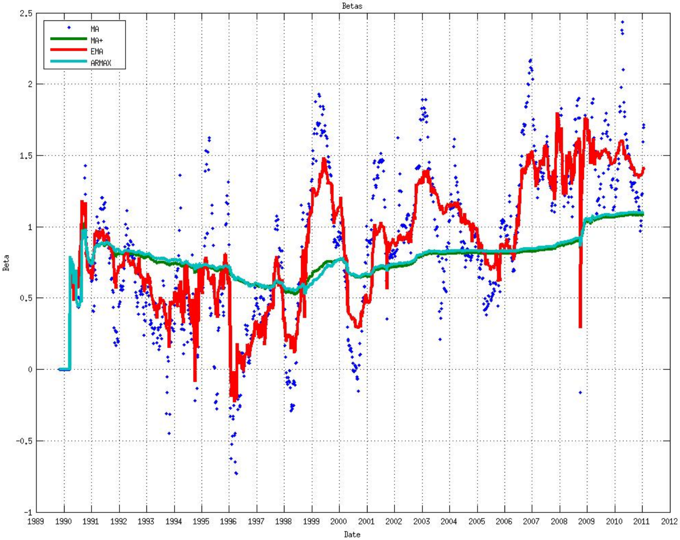
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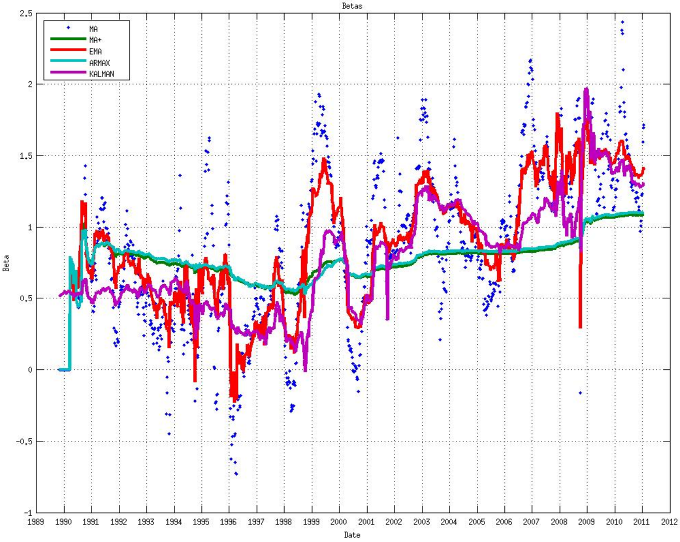
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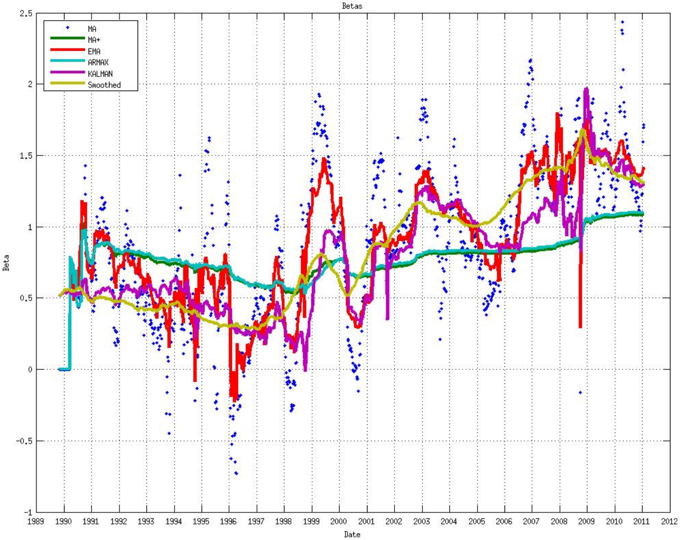
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What use in the Equity world ?

Objectives

- Forecasting returns
- Dimension réduction
- Cost efficient replication
- Leverage synthetic replication

Factors

- Subset of the equity world
- Econometric data
- Derivates (Futures)

Model

State Space

Define α_t as the hidden state at time t

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$$\text{Measurement} \qquad \underset{(n \times 1)}{y_t} = \qquad Z_t \alpha_t + d_t + \epsilon_t$$

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$$\text{Measurement} \quad \underset{(n \times 1)}{y_t} = \quad Z_t \alpha_t + d_t + \epsilon_t$$

$$\text{Transition} \quad \underset{(m \times 1)}{\alpha_t} = \quad T_t \alpha_{t-1} + c_t + R_t \eta_t$$

Where

$$\begin{pmatrix} \epsilon_t \\ \eta_t \end{pmatrix} \sim \mathcal{N}\left(0, diag(H_t, Q_t)\right)$$

$$\mathbb{E}\left[\alpha_0 \eta_t'\right] = 0$$

$$\mathbb{E}\left[\alpha_0 \epsilon_t'\right] = 0$$

Used for

- Prediction
- Filtering
- Smoothing

Standard ARMA(p,q)

$$y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}$$

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$$\begin{aligned} \Phi(L) y_t &= \theta(L) \epsilon_t \\ \epsilon_t &\sim \mathcal{N}(0, \sigma^2) \end{aligned}$$

where :

$$\begin{aligned} \Phi(L) &= (1 - \phi_1 L - \cdots - \phi_p L^p) \\ \Theta(L) &= (1 + \theta_1 L + \cdots + \theta_q L^q) \end{aligned}$$

ARMA(p,q) in State Space Form

Setting $m = \max(p, q + 1)$, the ARMA$(m, m - 1)$:

$$\begin{aligned}(\phi_1, \phi_2, \ldots, \phi_m) &= (\phi_1, \ldots, \phi_p, 0, \ldots, 0)\\(\theta_1, \theta_2, \ldots, \theta_{m-1}) &= (\theta_1, \ldots, \theta_q, 0, \ldots, 0)\end{aligned}$$

ARMA(p,q) in State Space Form

$$y_t = (1, 0, \ldots, 0)\alpha_t$$

$$\alpha_t = \left[\begin{array}{c|c} \begin{matrix}\phi_1 \\ \phi_2 \\ \vdots\end{matrix} & I_{m-1} \\ \hline \phi_m & 0' \end{array}\right] \alpha_{t-1} + \begin{bmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_{m-1} \end{bmatrix} \epsilon_t$$

Evaluation

Initialization

$$a_0 = \mathbb{E}(\alpha_0)$$

Initialization

$$
\begin{aligned}
a_0 &= \mathbb{E}(\alpha_0) \\
P_0 &= \mathbb{E}[(\alpha_0 - a_0)(\alpha_0 - a_0)']
\end{aligned}
$$

Prediction

Dropping c_t and d_t :

$$a_{t|t-1} = T_t a_{t-1}$$

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Define :

$$\text{errors } v_t = y_t - \hat{y}_t$$

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Define :

$$\begin{aligned} \text{errors } v_t &= y_t - \hat{y}_t \\ \text{errors}' \text{ variance } F_t &= T_t P_{t|t-1} Z_t' + H_t \end{aligned}$$

Correction

Updating State estimate a_{t-1} after measurement

$$a_t = a_{t|t-1} + K_t v_t$$

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Minimize MSE on State yields the Kalman gain K_t is :

$$K_t = P_{t|t-1} Z' F_t^{-1}$$

Estimation

- Inital values may need some work. . .

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- $v_t \sim \mathcal{N}(0, F_t) \Rightarrow \ln(\mathcal{L})$

$$\ln(\mathcal{L}) = -\frac{1}{2}\sum_{t=1}^{\tau}\left[n \ln 2\pi + \ln|F_t| + v_t' F_t^{-1} v_t\right]$$

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- Minimize $\ln(\mathcal{L})$

Application

- Macroeconomic

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- Interest rate

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- Portfolio management

Business Condition Index

Aruba, Dieblod and Scotti have built this index

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- Mutiple factors and frequencies
- High Frequency : daily
- Optimal estimation

Data

Business condition is dependent on :

- GDP (quarterly flow)

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- . . .

State

Business condition is a daily $AR(1)$ process

$$\alpha_t = T\alpha_{t-1} + \eta_t,\ \eta_t \sim \mathcal{N}(0,1)$$

Observed factors

Removing trends, business dependent factors can be modeled

$$y_t^i = \phi_i y_{t-\tau_i}^i + z_i \alpha_t + \epsilon_t^i,$$

- τ_i is period of measurement of y^i in days
- $\epsilon_t^i \sim \mathcal{N}(0, \sigma_i^2)$

Synchronizing

y_t^i is not measured daily, therefore define $\tilde{y}_t^i$

- in case of a *stock* :

$$\tilde{y}_t^i = \begin{cases} y_t^i & \text{if observed} \\ nan & \text{otherwise} \end{cases}$$

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- in case of a *flow* :

$$\tilde{y}_t^i = \begin{cases} \sum_{j=0}^{\tau_i - 1} y_{t-j}^i & \text{if observed} \\ nan & \text{otherwise} \end{cases}$$

$$\underbrace{\begin{bmatrix} \tilde{y}_t^1 \\ \tilde{y}_t^2 \\ \tilde{y}_t^3 \\ \tilde{y}_t^4 \end{bmatrix}}_{y_t} = \underbrace{\begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ 0 & z_2 & 0 & z_4 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & z_2 & 0 & z_4 \\ 0 & 0 & 0 & z_4 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & z_4 \end{bmatrix}'}_{Z_t} \underbrace{\begin{bmatrix} \alpha_t \\ \alpha_{t-1} \\ \vdots \\ \alpha_{t-q} \end{bmatrix}}_{\boldsymbol{\alpha}_t}$$

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$$\underbrace{\begin{bmatrix}\alpha_t \\ \alpha_{t-1} \\ \vdots \\ \alpha_{t-q}\end{bmatrix}}_{\boldsymbol{\alpha}_t} = \underbrace{\left[\begin{array}{cc} T & 0' \\ \hline I_q & 0\end{array}\right]}_{T_t} \underbrace{\begin{bmatrix}\alpha_{t-1} \\ \alpha_{t-2} \\ \vdots \\ \alpha_{t-q-1}\end{bmatrix}}_{\boldsymbol{\alpha}_{t-1}} + \underbrace{\begin{bmatrix}1\\0\\ \vdots \\ 0\end{bmatrix}}_{R_t} \eta_t, \; \begin{pmatrix}\epsilon_t \\ \eta_t\end{pmatrix} \sim \mathcal{N}\left(0, diag\begin{bmatrix}\sigma_1^2\\ \sigma_2^2\\ \sigma_3^2\\ \sigma_4^2\\ 1\end{bmatrix}\right)$$

Estimation

There will be 3 situations concerning missing values on day t :

- All : Prediction : $a_{t|t-1}$ and $P_{t|t-1}$

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Initialization is done in 2 steps :

- Use only *stock* factors to reduce the size and get a_t
- Use first step results to regress and get the remaining start-up values

Results

An online daily calculated *Business Condition Indicator* with a few more refinements

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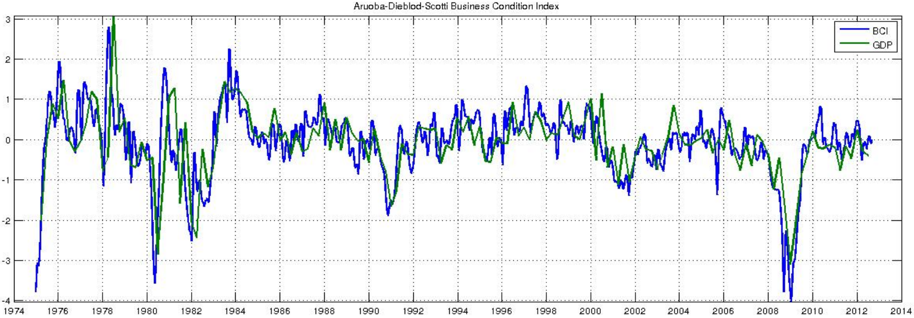


Available at *Federal Reserve Bank of Philadelphia*

Interest rate curve

Objectives

Dimension reduction

Get a global interest rate curve

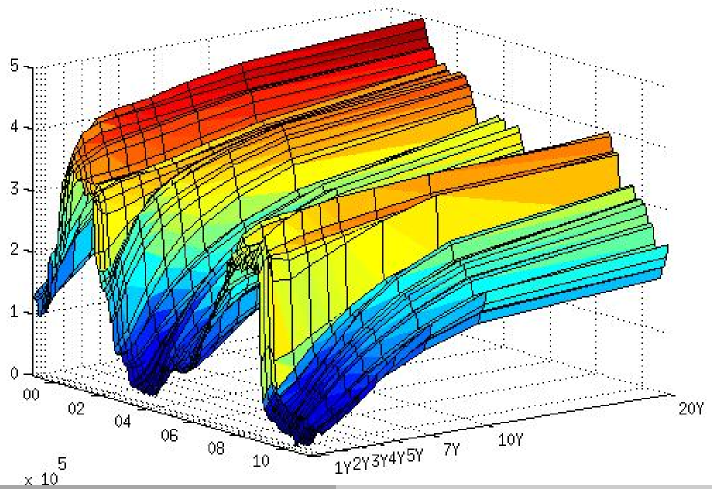
Yield Curve

We need the curve to price interest rate products.
Define :

- Maturity τ in days
- The interest rate at time t for the maturity τ as $y_t(\tau)$

If at time t you invest 1 CHF for τ days, you get $(1 + y_t(\tau))^\tau$ CHF

Yield Curve



Data

The tenors : 11 maturities

- $\tau \leq 1Y$: Libor 3M, 6M, 9M and 12M
- $\tau > 1Y$: Interest Rate Swap 2Y,3Y,4Y,5Y,7Y,10Y and 20Y

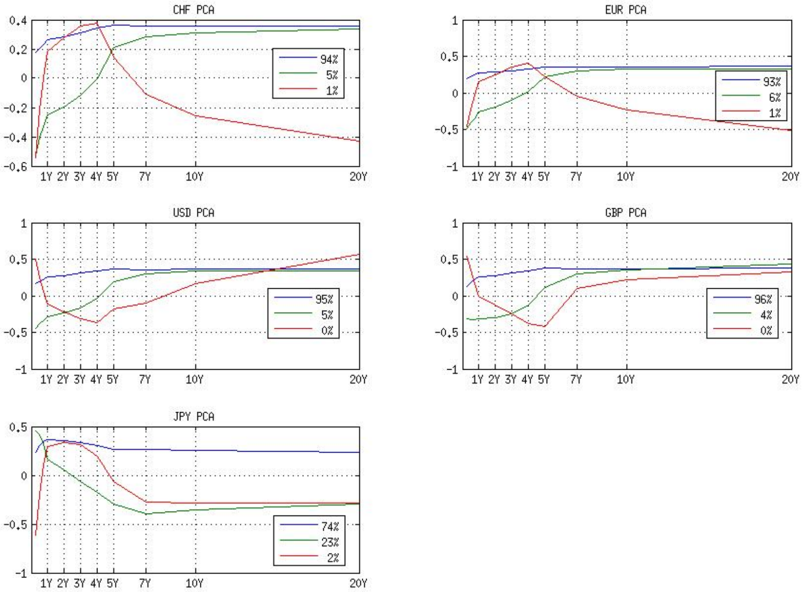
The currencies : CHF, EUR, USD, GBP, JPY

Size reduction

To reduce size, one looks at factors such as :

- Level : $y_t(10Y)$
- Slope : $y_t(10Y) - y_t(3M)$
- Curvature : $2y_t(2Y) - y_t(10Y) - y_t(3M)$

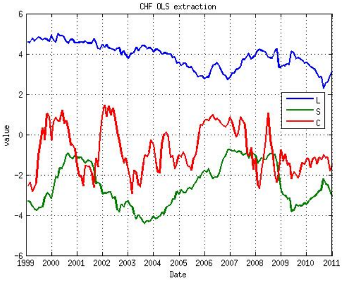
Size reduction

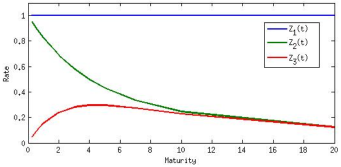


Curve Model

Nelson-Siegel proposed a good fitting model

$$\begin{aligned} \text{Level}: \quad Z_t^1(\tau) &= 1 \\ \text{Slope}: \quad Z_t^2(\tau) &= \frac{1-e^{-\lambda\tau}}{\lambda\tau} \\ \text{Curvature}: \quad Z_t^3(\tau) &= \frac{1-e^{-\lambda\tau}}{\lambda\tau} \end{aligned}$$





State Space Form

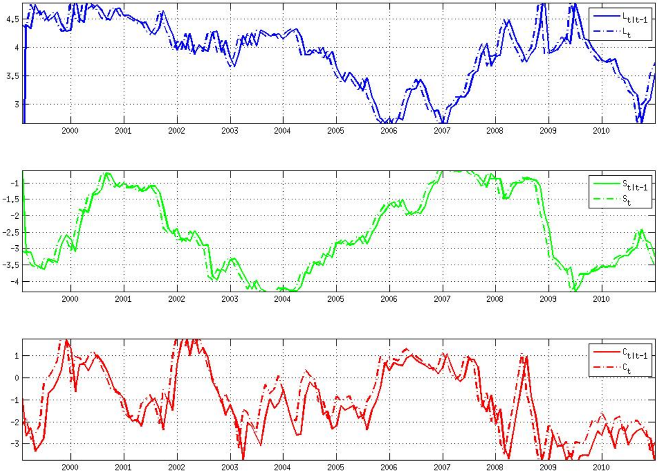
$$\begin{bmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_n) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_n}}{\lambda\tau_n} & \frac{1-e^{-\lambda\tau_n}}{\lambda\tau_n} - e^{-\lambda\tau_n} \end{bmatrix} \begin{bmatrix} l_t \\ s_t \\ c_t \end{bmatrix} + \begin{bmatrix} \epsilon_t(\tau_1) \\ \epsilon_t(\tau_2) \\ \vdots \\ \epsilon_t(\tau_n) \end{bmatrix}$$

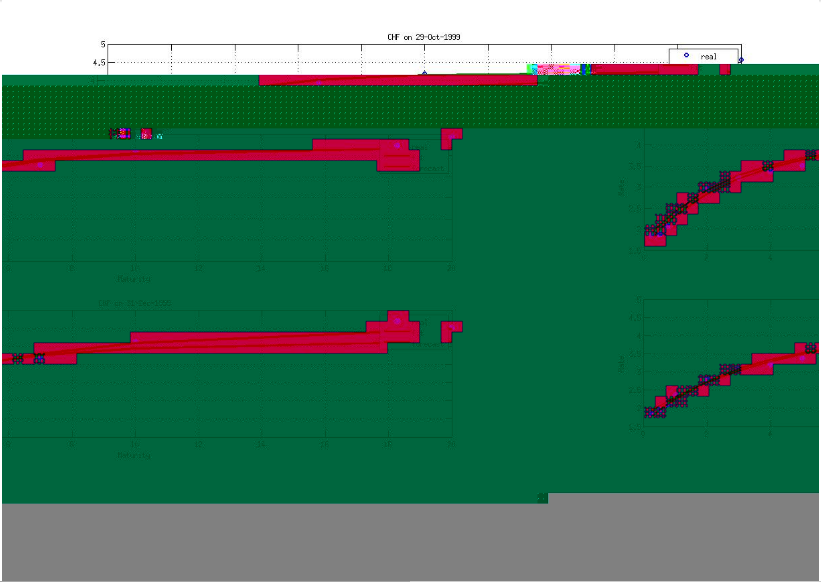
State Space Form

$$\begin{bmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_n) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_n}}{\lambda\tau_n} & \frac{1-e^{-\lambda\tau_n}}{\lambda\tau_n} - e^{-\lambda\tau_n} \end{bmatrix} \begin{bmatrix} l_t \\ s_t \\ c_t \end{bmatrix} + \begin{bmatrix} \epsilon_t(\tau_1) \\ \epsilon_t(\tau_2) \\ \vdots \\ \epsilon_t(\tau_n) \end{bmatrix}$$

Or with $\alpha_t = (l_t s_t c_t)'$

$$\begin{aligned} y_t &= Z(\tau)\alpha_t + \epsilon_t \\ \alpha_t &= T\alpha_{t-1} + \eta_t \\ \begin{bmatrix} \epsilon_t \\ \eta_t \end{bmatrix} &\sim \mathcal{N}\left(0, \begin{bmatrix} H & 0 \\ 0 & Q \end{bmatrix}\right) \end{aligned}$$


4,5
4
3,5
3
2000
2001
2002
2003
2004
2005
2006
2007
2008
2009
2010
$L_{t|t-1}$
L_t
-1
-1,5
-2
-2,5
-3
-3,5
-4
$S_{t|t-1}$
S_t
1
0
-1
-2
-3
$C_{t|t-1}$
C_t


CHF on 29-Oct-1999
real
CHF on 31-Dec-1999
Maturity

Global Model

We start by removing the *Curvature* Factor C_t for simplicity

$$Y_t = L_t + \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau}\right) S_t + \epsilon_t(\tau)$$

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$$\begin{bmatrix} L_t \\ S_t \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{22} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} L_{t-1} \\ S_{t-1} \end{bmatrix} + \begin{bmatrix} U_t^L \\ U_t^S \end{bmatrix}$$

$$U_t \sim \mathcal{N}\left(0, I_2\right)$$

Country Model

Each country's curve (l_{it}, s_{it}) are linked to the global curve, $i = 1, \ldots, N$

$$
\begin{aligned}
l_{it} &= \alpha_i^l + \beta_i^l L_t + \gamma_{it}^l \\
s_{it} &= \alpha_i^s + \beta_i^s S_t + \gamma_{it}^s
\end{aligned}
$$

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$$\begin{aligned} \begin{bmatrix} \gamma_{it}^l \\ \gamma_{it}^s \end{bmatrix} &= \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \begin{bmatrix} \gamma_{it-1}^l \\ \gamma_{it-1}^s \end{bmatrix} + \begin{bmatrix} u_{it}^l \\ u_{it}^s \end{bmatrix} \\ \mathbb{E} u_{it}^n u_{i't'}^{n'} &= \delta_{ii'} \delta_{tt'} \delta_{nn'} \left(\sigma_i^n\right)^2, \; n = s, l \\ \mathbb{E} u_{it}^n U_{t-j}^{n'} &= 0, \forall\, i, t, n, j \end{aligned}$$

State Space

$$\begin{bmatrix} y_{1t}(\tau_1) \\ y_{1t}(\tau_2) \\ \vdots \\ y_{Nt}(\tau_n) \end{bmatrix} = A \begin{bmatrix} \alpha_1^l \\ \alpha_1^s \\ \vdots \\ \alpha_N^s \end{bmatrix} + B \begin{bmatrix} L_t \\ S_t \end{bmatrix} + A \begin{bmatrix} \gamma_{1t}^l \\ \gamma_{1t}^s \\ \vdots \\ \gamma_{Nt}^s \end{bmatrix} + \begin{bmatrix} \epsilon_{1t}(\tau_1) \\ \epsilon_{1t}(\tau_2) \\ \vdots \\ \epsilon_{Nt}(\tau_n) \end{bmatrix}$$

State Space

$$A = \begin{bmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & 0 & \cdots & 0 \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & 1 & \frac{1-e^{-\lambda\tau_n}}{\lambda\tau_n} \end{bmatrix} \qquad B = \begin{bmatrix} \beta_1^l & \beta_1^s \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} \\ \beta_1^l & \beta_1^s \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} \\ \vdots & \vdots \\ \beta_N^l & \beta_N^s \frac{1-e^{-\lambda\tau_n}}{\lambda\tau_n} \end{bmatrix}$$

Estimation

MCMC :

- Get $Z = (l_{it}, s_{it})$ by OLS and use them as observed to estimate the remaining parameters : $\theta = (\alpha_i^n, \beta_i^n, \psi, \sigma_i^n, \phi)$, $n = l, s$

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- Get $F = (L_t, S_t)$ by drawing interatively from $\theta^{(i)} \sim p(\theta | F^{(i-1)}, Z)$ and from $F^{(i)} \sim p(F|\theta^{(i)}, Z)$

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- ...

Appealing Model...

Appealing Model...

...but can become delicate to estimate

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If little is know from the underlying dynamic